

WHY SUMS OF DISTANCES LEAD TO PI

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ABSTRACT. A simple calculation with consecutive integers gives rational numbers approaching π : add the distances from each integer to the others, take reciprocals, add them, and multiply by the size of the group. I explain the limit using a Riemann sum and a picture of a right angle. The same picture shows that every approximation is larger than π .

1. THE OBSERVATION

In a Reddit post, u/Adventurous_Can_7236 grouped the positive integers into rows of lengths $1, 2, 3, \dots$ and noticed an unexpected appearance of π [Ad]. For each entry in a row, add its distances to all the entries in that row. Take the reciprocals of these sums, add them, and multiply by the number of entries.

For example, the fourth row is $7, 8, 9, 10$. Its distance sums are $6, 4, 4, 6$, so the result is

$$4 \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} \right) = \frac{10}{3} = 3.333333 \dots$$

Moving the whole row six places to the left gives $1, 2, 3, 4$ and preserves every distance. In general, any row of n consecutive integers gives the same answer as $1, \dots, n$.

	7	8	9	10
entry				
distances	$0 + 1 + 2 + 3$	$1 + 0 + 1 + 2$	$2 + 1 + 0 + 1$	$3 + 2 + 1 + 0$
sum	6	4	4	6
reciprocal	$1/6$	$1/4$	$1/4$	$1/6$

FIGURE 1. The fourth row. Add the bottom entries and multiply by 4.

For integers $n \geq 2$ and $1 \leq k \leq n$, write

$$(1) \quad D_n(k) = \sum_{j=1}^n |k - j|, \quad S_n = n \sum_{k=1}^n \frac{1}{D_n(k)}.$$

Thus S_n is the number produced by the observation. The row of length 1 is excluded because its distance sum is 0.

Theorem 🍌. *For every integer $n \geq 2$, we have*

$$S_n > \pi, \quad \lim_{n \rightarrow \infty} S_n = \pi.$$

The comments on the post identify a Riemann sum and the distance array in OEIS A144216 [Ki]. I give the calculation in Section 3 and an angle interpretation in Section 4. It is a nice example of a circle appearing in a question about integers.

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2. COUNTING THE DISTANCES

The distances from k to the entries on its left are $1, \dots, k-1$, and those to the entries on its right are $1, \dots, n-k$. Using $1 + \dots + r = r(r+1)/2$ gives

$$(2) \quad D_n(k) = \frac{k(k-1)}{2} + \frac{(n-k)(n-k+1)}{2}.$$

These are two triangular numbers. Notice that $D_n(k) = D_n(n+1-k)$, which explains the symmetry of each row.

Substituting eq. (2) into eq. (1) gives the explicit formula

$$(3) \quad S_n = 2n \sum_{k=1}^n \frac{1}{k(k-1) + (n-k)(n-k+1)}.$$

Near the ends of a long row, the distance sum is about $n^2/2$. At its middle, it is about $n^2/4$, so the reciprocals are larger there. This curved profile is what produces π .

3. FROM A SUM TO AN AREA

To see the profile clearly, place the n entries at the midpoints of n equal intervals in $[-1, 1]$. Their coordinates are

$$x_k = -1 + \frac{2k-1}{n} \quad (1 \leq k \leq n).$$

Completing the square in eq. (2) gives an exact identity:

$$D_n(k) = \left(k - \frac{n+1}{2}\right)^2 + \frac{n^2-1}{4} = \frac{n^2}{4} \left(1 + x_k^2 - \frac{1}{n^2}\right).$$

Therefore

$$(4) \quad S_n = \sum_{k=1}^n \underbrace{\frac{2}{n}}_{\text{width}} \underbrace{\frac{2}{1 + x_k^2 - 1/n^2}}_{\text{height}}.$$

This is exactly the total area of n rectangles. As n increases, the rectangles become narrower and their heights approach the curve $f(x) = 2/(1+x^2)$.

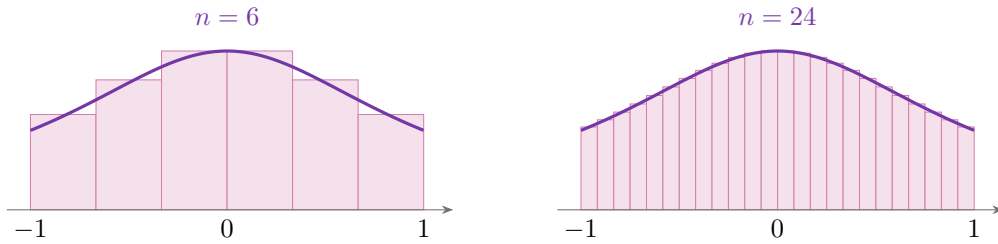


FIGURE 2. Pink rectangles have total area S_n . The purple curve is $y = 2/(1+x^2)$, whose area over $[-1, 1]$ is π .

The height $2/(1+x^2 - 1/n^2)$ differs from $f(x)$ by a number between 0 and $2/(n^2-1)$, uniformly for $-1 \leq x \leq 1$. Replacing the heights in eq. (4) by $f(x_k)$ therefore changes the total area by at most $4/(n^2-1)$. Since the midpoint Riemann sums of the continuous function f converge to its integral, we obtain

$$(5) \quad \lim_{n \rightarrow \infty} S_n = \int_{-1}^1 \frac{2}{1+x^2} dx = 2(\arctan(1) - \arctan(-1)) = \pi.$$

The appearance of π comes from $\arctan(1) = \pi/4$, the angle of a right triangle with equal legs. Equivalently, substituting $x = \tan \theta$ changes the integral into $2 \int_{-\pi/4}^{\pi/4} d\theta$.

4. A RIGHT ANGLE HIDDEN IN THE SUM

There is also an exact angle interpretation before taking the limit. Join the origin to the endpoints of the n equal intervals on the horizontal segment from $(-1, 1)$ to $(1, 1)$, as in fig. 3. The rays cut a right angle into n smaller angles. Writing these angles as

$$\theta_k = \arctan\left(x_k + \frac{1}{n}\right) - \arctan\left(x_k - \frac{1}{n}\right),$$

we have $0 < \theta_k < \pi/2$ and

$$\sum_{k=1}^n \theta_k = \frac{\pi}{2}.$$

The subtraction formula for the tangent gives

$$(6) \quad \tan \theta_k = \frac{2/n}{1 + x_k^2 - 1/n^2}, \quad S_n = 2 \sum_{k=1}^n \tan \theta_k.$$

Thus the original procedure adds twice the tangents of angles that together make a right angle.

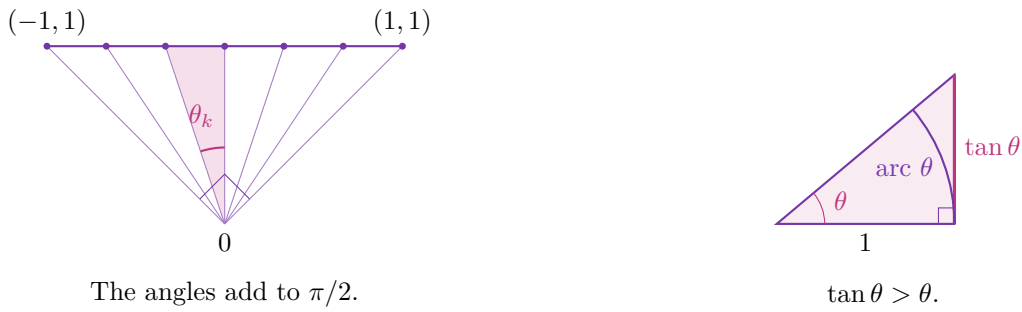


FIGURE 3. Left: equal intervals give angles whose sum is a right angle. Right: on a unit circle, the tangent segment is longer than the corresponding arc. Angles are measured in radians.

For $0 < \theta < \pi/2$, the inequality $\tan \theta > \theta$ follows from

$$\tan \theta - \theta = \int_0^\theta \tan^2 t \, dt > 0.$$

Applying it term by term in eq. (6) gives $S_n > 2 \sum_k \theta_k = \pi$, proving the remaining assertion of Theorem 🍌. Since $(\arctan x)' \leq 1$, every θ_k is at most $2/n$. The approximation $\tan \theta / \theta \rightarrow 1$ as $\theta \rightarrow 0$ then also proves the limit directly: all the small angles shrink, while their total remains $\pi/2$.

5. HOW CLOSE IS THE ANSWER?

The numerical values approach π quite quickly. In the last column, the values suggest that the error is proportional to $1/n^2$.

n	S_n	$n^2(S_n - \pi)$
2	4.000000000000	3.433629
4	3.333333333333	3.067851
10	3.170887016048	2.929436
100	3.141883091672	2.904381
1000	3.141595557722	2.904132

In fact, the constant in the last column has a simple value.

Proposition 5.1. *As $n \rightarrow \infty$, we have*

$$S_n = \pi + \frac{\pi/2 + 4/3}{n^2} + O(n^{-4}).$$

Proof. By eq. (6), $\tan \theta_k = O(n^{-1})$ uniformly in k . Expanding $u - \arctan u = u^3/3 + O(u^5)$ and using $\sum_k \theta_k = \pi/2$ gives

$$\begin{aligned} S_n - \pi &= \frac{2}{3} \sum_{k=1}^n (\tan \theta_k)^3 + O(n^{-4}) \\ &= \frac{16}{3n^2} \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{(1+x_k^2)^3} \right) + O(n^{-4}). \end{aligned}$$

The replacement of $1+x_k^2 - 1/n^2$ by $1+x_k^2$ contributes $O(n^{-4})$. Taylor's theorem on each interval shows that the midpoint sum in parentheses differs from half the integral by $O(n^{-2})$, because the second derivative of $(1+x^2)^{-3}$ is bounded on $[-1, 1]$. Finally, the substitution $x = \tan t$ gives

$$\int_{-1}^1 \frac{dx}{(1+x^2)^3} = \int_{-\pi/4}^{\pi/4} \cos^4 t \, dt = \frac{1}{2} + \frac{3\pi}{16}.$$

Multiplication by $8/3$ gives $4/3 + \pi/2$, as claimed. $\square$

Thus multiplying the row length by 10 makes the error about 100 times smaller. For $n = 1000$, the excess is about 0.000002904, which explains the agreement with π in the original observation.

REFERENCES

- [Ad] u/Adventurous_Can_7236, *i found a pi formula*, r/mathematics, Reddit, 15 October 2026. [Post and discussion](#).
- [Ki] C. Kimberling, *A144216: $\binom{m}{2} + \binom{n}{2}$, $m, n \geq 1$* , The On-Line Encyclopedia of Integer Sequences, 2008. <https://oeis.org/A144216>. Accessed 15 October 2026.