

WHEN A MOY–PRASAD SUBGROUP OF GL_n IS UNIFORM

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ABSTRACT. For p -adic Moy–Prasad subgroups of GL_n above depth $1/(p-1)$, I give an exact uniformity test. Logarithms supply the roots, and commutators do the rest. Depth one for odd primes and two for $p=2$ are sharp universal bounds, as 3×3 examples show. Uniformity can come and go as the depths grow.

1. INTRODUCTION

For odd p , Perdue [Pe] constructs a saturated p -valuation on the Moy–Prasad subgroup $G_{x,1/(p-1)+}$ of a connected reductive group with a weakly induced maximally split maximal torus, and deduces that $G_{x,r}$ is a uniform pro- p group whenever $r \geq 1$ (see [Pe, Corollary 5.8], and [KOS, Corollary C.4] for the torsion-free case with $p \geq 5$). I had been interested in this uniformity question for some time, and Perdue’s nice paper inspired me to finally write this note. In the following I look at the range $1/(p-1) < r < 1$ for the general linear group, where the answer turns out to be a nice little puzzle about the exponents in a matrix.

Let $F/\mathbb{Q}_p$ be finite, with ramification index e , uniformizer ϖ and valuation normalized by $v(p) = 1$. Fix real numbers $s_1, \dots, s_n$ and set

$$(1.1) \quad w(A) = \min_{1 \leq i, j \leq n} (v(a_{ij}) + s_i - s_j), \quad A = (a_{ij}) \in M_n(F),$$

where $v(0) = \infty$. For $r > 0$, let $G_{s,r}$ be the group of all $g \in \mathrm{GL}_n(F)$ with $w(g-1) \geq r$. Concretely, a matrix g lies in $G_{s,r}$ exactly when the (i, j) entry of $g-1$ lies in $\varpi^{m_{ij}} \mathcal{O}_F$ for all i, j , where

$$(1.2) \quad m_{ij} = \lceil e(r - s_i + s_j) \rceil.$$

These are the Moy–Prasad subgroups of $\mathrm{GL}_n(F)$ at the points of the standard apartment, as recalled in Section 3, and every point of the building lies in a conjugate of this apartment.

Here is a small example that shows what can happen. Take $F = \mathbb{Q}_3$, $s = (0, 1/4, 1/2)$ and $r = 3/4$. Then

$$G_{s,3/4} = 1 + \begin{pmatrix} 3\mathbb{Z}_3 & 3\mathbb{Z}_3 & 9\mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 3\mathbb{Z}_3 & 3\mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 3\mathbb{Z}_3 & 3\mathbb{Z}_3 \end{pmatrix}.$$

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Notice that this group contains the matrices

$$g = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad ghg^{-1}h^{-1} = \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, every element $1 + A$ of this group has $(1 + A)^3 = 1 + 3A + 3A^2 + A^3$, which is congruent to 1 modulo 9 and has its $(1, 3)$ entry divisible by 27. The matrices with these two properties form a closed subgroup. It contains all cubes, but not the commutator of g and h . For odd p , a uniform pro- p group is powerful, i.e. every commutator lies in the closed subgroup generated by p th powers (see [DDMS, Chapter 4]). So $G_{s,3/4}$ is not uniform, although its depth $3/4$ is larger than $1/(p-1) = 1/2$ and it carries Perdue's saturated 3-valuation. The first theorem says exactly when this happens.

Theorem 🍌. *Let p be odd and $r > 1/(p-1)$. Then*

$$G_{s,r} \text{ is uniform} \iff m_{ij} + m_{jk} \geq m_{ik} + e \text{ for all } i, j, k \text{ not all equal.}$$

Over $\mathbb{Q}_p$ and for $r \leq 1$, all diagonal exponents m_{ii} are equal to 1, and the condition becomes $m_{ij} + m_{jk} > m_{ik}$ for all i, j, k with $i \neq j$ and $j \neq k$. In words, going from i to k through j must cost strictly more than going directly. The path $1 \rightarrow 2 \rightarrow 3$ in the example above costs only $m_{12} + m_{23} = 2 = m_{13}$. For $r \geq 1$ the condition always holds, which is the result of Perdue for GL_n .

The second theorem shows that the bound $r \geq 1$ cannot be lowered, at any depth.

Theorem 🍌. *Let p be odd and $0 < r < 1$. Then $G_{s,r} \subseteq \text{GL}_3(F)$ is not uniform for*

$$s = (0, 1 - r, 2 - 2r).$$

For $r > 1/(p-1)$ this follows from Theorem 🍌, since $m_{12} = m_{23} = e$ and $m_{13} > e$. The example above is the case $F = \mathbb{Q}_3$ and $r = 3/4$. In Section 3, the proof works for every $0 < r < 1$ with one commutator. Over $\mathbb{Q}_p$, three by three matrices are really needed here. For 2×2 matrices over $\mathbb{Q}_p$, every $G_{s,r}$ with $r > \max\{1/2, 1/(p-1)\}$ is uniform, while the pro- p Iwahori subgroup at depth $1/2$ is not (see Example 3.5). Also, uniformity at one depth does not imply uniformity at a larger depth. For $p = 5$ and $s = (0, 1/5, 2/5)$, the group $G_{s,r}$ is uniform for $2/5 < r \leq 3/5$, not uniform for $3/5 < r \leq 4/5$, and uniform again for $r > 4/5$ (see Figure 1).

For $p = 2$, the same calculation replaces e by $2e$ in the criterion and gives the sharp universal bound $r \geq 2$ (see Proposition 3.3).

The proof of Theorem 🍌 uses that each element of $G_{s,r+1}$ has a p th root in $G_{s,r}$. In Section 2 I prove this with matrix logarithms in a slightly more general form, for intersections with algebraic subgroups and for every prime (Theorem 2.1). The estimates behind it are classical. Lazard treats valued associative algebras and weighted matrix groups in [La, III, Sections 3.2.6–3.2.7], and saturation for standard formal groups, without excluding the prime 2, is proved in [HKN, Lemma 2.2.2]. I make no claim that the general matrix saturation theorem is new. In Section 4 the same argument becomes a two-line computation for the unitary root group of [Pe, Section 3.2], with an explicit root formula. Finally, Section 5 corrects the ramification rounding in a torus descent argument and two root-vector formulas in the arXiv version [Pe].

2. THE MATRIX ARGUMENT

In this section I show that the weighted matrix groups, and their intersections with algebraic subgroups, carry saturated p -valuations. Here p may be any prime. If $H \subseteq GL_n$ is a closed algebraic F -subgroup, put

$$H_{>1/(p-1)} = \{g \in H(F) : w(g-1) > 1/(p-1)\}.$$

A p -valuation on a group is a function ω , finite away from 1 and with $\omega(1) = \infty$, such that

$$\begin{aligned} \omega(g) &> \frac{1}{p-1}, & \omega(g^{-1}h) &\geq \min\{\omega(g), \omega(h)\}, \\ \omega([g, h]) &\geq \omega(g) + \omega(h), & \omega(g^p) &= \omega(g) + 1. \end{aligned}$$

Here $[g, h] = ghg^{-1}h^{-1}$. A complete p -valued group is saturated if every element of valuation greater than $p/(p-1)$ is a p th power.

Theorem 2.1. *The group $H_{>1/(p-1)}$ is saturated for the p -valuation $\omega(g) = w(g-1)$. If $\omega(g) > p/(p-1)$, its unique p th root in this group is*

$$g^{1/p} = \exp(p^{-1} \log g).$$

The weights may be irrational, and H need not be reductive or connected. Here the main point is that the displayed root satisfies the equations of H . In [Section 4](#) I apply the argument to the matrix subgroup used in [\[Pe, Section 3.2\]](#). At $p = 2$ this proves saturation of that explicitly defined matrix subgroup. Identifying it with a canonical Moy-Prasad subgroup is a separate question and is not needed here.

For the proof, I show that logarithms preserve depth and that analytic powers preserve algebraic equations. The elementary inequalities

$$(2.1) \quad w(A+B) \geq \min\{w(A), w(B)\}, \quad w(AB) \geq w(A) + w(B), \quad w(tA) = v(t) + w(A)$$

follow directly from [\(1.1\)](#). This valuation defines the usual topology on $M_n(F)$. Its finite values lie in a finite union of translates of $v(F^\times)$, so every strict depth condition defines a closed lattice.

Lemma 2.2. *If $w(A) > 1/(p-1)$, then*

$$w(\log(1+A)) = w(A), \quad w(\exp A - 1) = w(A).$$

The usual logarithm and exponential are mutually inverse on these neighborhoods.

Proof. For every integer $k \geq 2$, one has $v_p(k) \leq (k-1)/(p-1)$ and $v_p(k!) \leq (k-1)/(p-1)$. Therefore

$$w(A^k/k) > w(A), \quad w(A^k/k!) > w(A),$$

and both valuations tend to infinity as k tends to infinity. The first term has strictly smaller valuation than the remaining terms, with a positive gap, which proves the two equalities. Mutual invertibility follows from the formal one-variable identities, since all the series involved converge. $\square$

Lemma 2.3. *Let $g \in H(F)$ satisfy $w(g-1) > 1/(p-1)$. If $t \in F$ and*

$$v(t) + w(\log g) > \frac{1}{p-1},$$

then $\exp(t \log g) \in H(F)$.

Proof. Write $X = \log g$. After clearing powers of the determinant, the equations of H inside GL_n can be written as polynomial equations in the matrix entries. For each such polynomial f , the function $f(\exp(tX))$ is a convergent power series on the disk given by the displayed inequality. It vanishes at every $t = p^k$ with $k \geq 0$, because $\exp(p^k X) = g^{p^k} \in H(F)$. These points tend to zero. A convergent power series with such zeros is identically zero, since otherwise its first nonzero term dominates sufficiently near zero. It follows that all the defining equations vanish wherever this series converges. The exponential is invertible, with inverse $\exp(-tX)$, so it lies in $H(F)$. $\square$

Proof of Theorem 2.1. The set of matrices $1 + A$ with $w(A) > 1/(p-1)$ is a group. Multiplication follows from (2.1), and inversion follows from the geometric series. For such a matrix, $w((1+A)^{-1}) = 0$. The identity

$$(1+A)^{-1}(1+B) - 1 = (1+A)^{-1}(B-A)$$

gives the second p -valuation inequality. Also,

$$[1+A, 1+B] - 1 = (AB - BA)(1+A)^{-1}(1+B)^{-1},$$

which gives the commutator inequality. By Lemma 2.2,

$$w(g^p - 1) = w(\exp(p \log g) - 1) = 1 + w(\log g) = 1 + w(g - 1).$$

Thus the restriction to $H_{>1/(p-1)}$ is a p -valuation.

The matrix depth condition defines a compact set, and the equations of H define a closed subset of it. Completeness follows. The depth quotients are finite p -groups, since their successive graded pieces embed in additive residue-field matrix spaces, so the group is pro- p . It also has finite rank, as a closed subgroup of a compact p -adic analytic matrix group (see [DDMS]).

Suppose now that $w(g - 1) > p/(p-1)$. Then $w(p^{-1} \log g) > 1/(p-1)$, and Lemma 2.3 with $t = 1/p$ gives

$$h = \exp(p^{-1} \log g) \in H_{>1/(p-1)}, \quad h^p = g.$$

If $h_1^p = h_2^p$ in this group, applying logarithms gives $p \log h_1 = p \log h_2$, and hence $h_1 = h_2$. This proves saturation and the claimed root formula. $\square$

Notice that preserving the algebraic equations is essential. An arbitrary closed subgroup need not inherit saturation for the restricted valuation. For example, $1 + p^2 \mathbb{Z}_p$ with $\omega(g) = v_p(g-1)$ is not saturated when p is odd, since $1 + p^2$ has valuation $2 > p/(p-1)$ but its p th root has valuation 1 and lies outside this subgroup.

The passage from rational to irrational weights also has a short abstract form. This observation strengthens the limit statement in [Pe, Lemma 2.13].

Lemma 2.4. *Suppose saturated p -valuations ω_n on one group converge pointwise to a complete p -valuation ω . Then ω is saturated.*

Proof. If $\omega(g) > p/(p-1)$, choose n with $\omega_n(g) > p/(p-1)$. Saturation of ω_n supplies a p th root in the same group, which proves the required condition for ω . $\square$

3. WHEN THE GROUPS ARE UNIFORM

In this section I prove [Theorems](#) [🍌](#) and [🍌](#) and look at some examples. First, the root and torus descriptions in [\[Pe, Remark 2.17, Notation 4.6 and Section 5\]](#) identify $G_{s,r}$ with the Moy–Prasad group at the point with coordinates s , for $r > 0$. One inclusion follows from its root generators $1 + tE_{ij}$, with $v(t) + s_i - s_j \geq r$, and its diagonal generators. Here E_{ij} is the usual matrix unit. For the reverse inclusion, Gaussian elimination has diagonal pivots of the form $1 + b$ with $v(b) > 0$. Its correction terms have weighted depth at least $2r \geq r$, so elimination preserves the entry bounds and produces the required root and diagonal factors. Recall also that a finitely generated pro- p group with p odd is uniform if and only if it is powerful and torsion-free (see [\[DDMS, Theorem 4.5\]](#)).

Commutators and p th powers in $G_{s,r}$ can be read off from the matrices directly.

Lemma 3.1. *Let $r > 0$, and let $g = 1 + A$ and $h = 1 + B$ be in $G_{s,r}$. Then*

$$w([g, h] - 1) = w(AB - BA), \quad w(g^p - 1) \geq \min\{w(A) + 1, pw(A)\}.$$

Proof. Notice that $[g, h] - 1 = (AB - BA)g^{-1}h^{-1}$, and that $g^{-1}h^{-1} = 1 + C$ with $w(C) \geq r > 0$ by the geometric series. So $[g, h] - 1$ is $AB - BA$ plus a matrix of strictly larger depth, unless both are zero. The second inequality follows from $g^p - 1 = \sum_{k=1}^p \binom{p}{k} A^k$, since p divides $\binom{p}{k}$ for $0 < k < p$. $\square$

Proposition 3.2. *Let p be odd and $r > 1/(p-1)$. Then*

$$G_{s,r} \text{ is uniform} \iff w(AB - BA) \geq r + 1 \text{ whenever } w(A) \geq r \text{ and } w(B) \geq r.$$

Proof. By [Theorem 2.1](#) for $H = \mathrm{GL}_n$, the group $G_{s,r}$ carries the p -valuation $\omega(g) = w(g-1)$. So it is a torsion-free pro- p group of finite rank, and it is uniform if and only if it is powerful. Since $pr \geq r + 1$, every p th power of an element of $G_{s,r}$ lies in $G_{s,r+1}$ by [Lemma 3.1](#). Conversely, every $g \in G_{s,r+1}$ satisfies $\omega(g) \geq r + 1 > p/(p-1)$. By [Theorem 2.1](#) and [Lemma 2.2](#), it is the p th power of $\exp(p^{-1} \log g)$, which has valuation $\omega(g) - 1 \geq r$. Therefore the p th powers of elements of $G_{s,r}$ are exactly the elements of the closed subgroup $G_{s,r+1}$, and $G_{s,r}$ is powerful if and only if all its commutators lie in $G_{s,r+1}$. By [Lemma 3.1](#), this is the stated condition. $\square$

Proof of Theorem [🍌](#). By (1.2), the matrices A with $w(A) \geq r$ are exactly the $\mathcal{O}_F$ -linear combinations of the matrices $\varpi^{m_{ij}} E_{ij}$. Using (2.1), it suffices to check the condition in [Proposition 3.2](#) for these basis matrices. The rule $E_{ij}E_{kl} = \delta_{jk}E_{il}$ shows that $XY - YX$, for two basis matrices X and Y , is a sum of at most two terms $\pm \varpi^{m_{ij}+m_{jk}} E_{ik}$ with i, j, k not all equal, in different entries. Each such term occurs, for $X = \varpi^{m_{ij}} E_{ij}$ and $Y = \varpi^{m_{jk}} E_{jk}$. Finally, the term $\varpi^{m_{ij}+m_{jk}} E_{ik}$ has depth $(m_{ij} + m_{jk})/e + s_i - s_k$. Since $m_{ij} + m_{jk}$ is an integer, this depth is at least $r + 1$ if and only if

$$m_{ij} + m_{jk} \geq \lceil e(r + 1 - s_i + s_k) \rceil = m_{ik} + e.$$

So the condition in [Proposition 3.2](#) is equivalent to the inequalities in [Theorem](#) [🍌](#). $\square$

Proof of Theorem [🍌](#). The matrices $g = 1 + pE_{12}$ and $h = 1 + pE_{23}$ have depth $1 + s_1 - s_2 = r$ and $1 + s_2 - s_3 = r$, so they lie in $G_{s,r}$. Their commutator is

$$[1 + pE_{12}, 1 + pE_{23}] = 1 + p^2 E_{13},$$

of depth $2 + s_1 - s_3 = 2r$. By [Lemma 3.1](#), every p th power in $G_{s,r}$ has depth at least $\min\{r+1, pr\}$, and the elements of at least this depth form a closed subgroup. Since $r < 1$ and $p \geq 3$, this minimum is larger than $2r$. So the commutator is not in the closed subgroup generated by p th powers. Hence $G_{s,r}$ is not powerful and cannot be uniform. $\square$

The same argument also gives a criterion at the prime 2. Recall that a finitely generated torsion-free pro-2 group is uniform if and only if it is powerful, which now means that every commutator lies in the closed subgroup generated by fourth powers (see [\[DDMS, Chapter 4\]](#)).

Proposition 3.3. *Let $p = 2$. For $r > 1$ we have*

$$G_{s,r} \text{ is uniform} \iff m_{ij} + m_{jk} \geq m_{ik} + 2e \text{ for all } i, j, k \text{ not all equal.}$$

All $G_{s,r}$ with $r \geq 2$ are uniform. In $\mathrm{GL}_3(F)$, one has

$$0 < r < 2, \quad s = (0, 2 - r, 4 - 2r) \implies G_{s,r} \text{ is not uniform.}$$

Proof. For $r > 1$, the group $G_{s,r}$ is torsion-free and pro-2 of finite rank by [Theorem 2.1](#). Its fourth powers form exactly $G_{s,r+2}$. To see this, [Lemma 2.2](#) gives $w(g^4 - 1) = w(g - 1) + 2$, and every $h \in G_{s,r+2}$ has the fourth root $\exp(\frac{1}{4} \log h)$ in $G_{s,r}$. By [Lemma 3.1](#), uniformity is therefore equivalent to $w(AB - BA) \geq r + 2$ whenever $w(A), w(B) \geq r$. Applying the matrix-unit calculation from the proof of [Theorem 2.1](#) gives the stated inequalities. They always hold for $r \geq 2$, since $w(AB - BA) \geq 2r \geq r + 2$.

For the stated weights and $0 < r < 2$, the matrices $1 + 4E_{12}$ and $1 + 4E_{23}$ have depth r , while their commutator $1 + 16E_{13}$ has depth $2r$. If $w(A) \geq r$, the binomial expansion gives

$$w((1 + A)^4 - 1) \geq \min\{r + 2, 2r + 1, 3r + 2, 4r\} > 2r.$$

The matrices of at least this depth form a closed subgroup containing every fourth power but excluding the commutator. So the group is not powerful and cannot be uniform. $\square$

Here are some examples for odd p . The first one shows that ramification matters.

Example 3.4. Let $r > 1/(p - 1)$. For $s = 0$ all exponents m_{ij} are equal to $\lceil er \rceil$, i.e. $G_{0,r} = 1 + \varpi^{\lceil er \rceil} M_n(\mathcal{O}_F)$. If $n \geq 2$, [Theorem 2.1](#) says that this group is uniform if and only if $\lceil er \rceil \geq e$, i.e. $r > 1 - 1/e$. This always holds over $\mathbb{Q}_p$. For $n \geq 2$ and any s , the triples (i, i, k) in [Theorem 2.1](#) show in the same way that $G_{s,r}$ is not uniform when $1/(p - 1) < r \leq 1 - 1/e$. Over $F = \mathbb{Q}_p(\sqrt{p})$ with $p \geq 5$, the group $1 + \sqrt{p} M_2(\mathcal{O}_F)$ has depth $1/2 > 1/(p - 1)$, but it is not uniform. To see this directly, notice that the commutator of $1 + \sqrt{p}E_{12}$ and $1 + \sqrt{p}E_{21}$ has diagonal entries $1 + p + p^2$ and $1 - p$, so its depth is $1 < 3/2$.

Example 3.5. For $s = (0, 1/2)$ and $0 < r \leq 1/2$, the group $G_{s,r}$ is the pro- p Iwahori subgroup

$$\begin{pmatrix} 1 + p\mathbb{Z}_p & p\mathbb{Z}_p \\ \mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}$$

of $\mathrm{GL}_2(\mathbb{Q}_p)$. Here $m_{21} + m_{12} = 1 = m_{11}$, and indeed

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - p & p^2 \\ -p & 1 + p + p^2 \end{pmatrix}$$

has depth 1, while every p th power has depth at least $3/2$ by [Lemma 3.1](#) applied at $r = 1/2$. So this group is not uniform for any odd p . For 2×2 matrices over $\mathbb{Q}_p$ and $r \leq 1$, the only

condition in [Theorem 3.5](#) that can fail is $m_{12} + m_{21} \geq 2$. Notice that $m_{12} + m_{21}$ is an integer at least $2r$. So every $G_{s,r} \subseteq GL_2(\mathbb{Q}_p)$ with $r > \max\{1/2, 1/(p-1)\}$ is uniform. Over a field with ramification index e and for $r \leq 1$, the conditions are $\lceil er \rceil \geq e$ and $m_{12} + m_{21} \geq \lceil er \rceil + e$, where $m_{12} + m_{21}$ can be as small as $\lceil 2er \rceil$. So all the groups $G_{s,r} \subseteq GL_2(F)$ with $r > 1/(p-1)$ are uniform if and only if $r > 1 - 1/(2e)$.

Example 3.6. Let $p = 5$ and $s = (0, 1/5, 2/5)$. The exponents m_{ij} only change when r crosses a multiple of $1/5$. [Figure 1](#) shows the four exponent matrices for $1/4 < r \leq 1$. In the first one, the path $3 \rightarrow 1 \rightarrow 3$ gives $m_{31} + m_{13} = 1 = m_{33}$, and in the third one the path $1 \rightarrow 2 \rightarrow 3$ gives $m_{12} + m_{23} = 2 = m_{13}$. For the other two matrices, every condition in [Theorem 3.5](#) holds. In particular, $G_{s,1/2} = 1 + 5M_3(\mathbb{Z}_5)$ is uniform, while its subgroup $G_{s,7/10}$ is not. For $r > 1$ the group is always uniform.

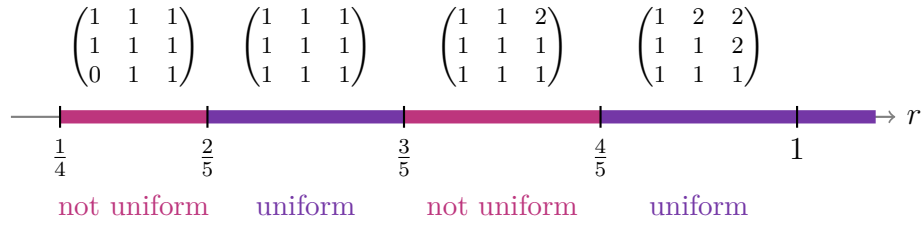


FIGURE 1. The exponent matrices (m_{ij}) for $p = 5$ and $s = (0, 1/5, 2/5)$, on the intervals $(1/4, 2/5]$, $(2/5, 3/5]$, $(3/5, 4/5]$ and $(4/5, 1]$. Purple means that $G_{s,r}$ is uniform, and pink means that it is not.

Remark 3.7. The first part of the proof of [Proposition 3.2](#) works for every group considered in [\[Pe\]](#). By [\[Pe, Corollary 5.8\(1\),\(2\)\]](#), the group $G_{x,r}$ with $r > 1/(p-1)$ is a torsion-free pro- p group, and its p th powers form $G_{x,r+1}$. It is also finitely generated, as a compact open subgroup of a p -adic Lie group. Therefore $G_{x,r}$ is uniform if and only if every commutator of two of its elements lies in $G_{x,r+1}$.

4. THE UNITARY ROOT GROUP

Let E/F be a quadratic extension of p -adic fields, and let $z \mapsto \bar{z}$ be its nontrivial automorphism. Normalize $v(p) = 1$ and consider

$$u(x, y) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & \bar{x} \\ 0 & 0 & 1 \end{pmatrix}, \quad y + \bar{y} = x\bar{x}.$$

This is the unitary root group of [\[Pe, Section 3.2\]](#). For $a \in \mathbb{R}$, put

$$\omega_a(u(x, y)) = \min\{v(x) + a, v(y) + 2a\}, \quad U^a = \{u(x, y) : \omega_a(u(x, y)) > 1/(p-1)\}.$$

The corresponding matrix weight is (1.1) with weights $(a, 0, -a)$.

Proposition 4.1. *For every prime p , the group (U^a, ω_a) is saturated, and*

$$(4.1) \quad \log u(x, y) = \begin{pmatrix} 0 & x & y - x\bar{x}/2 \\ 0 & 0 & \bar{x} \\ 0 & 0 & 0 \end{pmatrix}.$$

This logarithm preserves the weighted depth.

Proof. The formula follows from $(u(x, y) - 1)^3 = 0$. Its upper-right entry has trace zero because $y + \bar{y} = x\bar{x}$. Conversely, if $z + \bar{z} = 0$, then

$$(4.2) \quad \exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & \bar{x} \\ 0 & 0 & 0 \end{pmatrix} = u(x, z + x\bar{x}/2).$$

The quadratic correction has weighted valuation $2(v(x) + a) - v(2)$. On the stated domain this is greater than $v(x) + a$, since $v(2) = 0$ for odd p and $v(2) = 1$ for $p = 2$. Thus (4.1) and (4.2) preserve the minimum defining the depth.

The matrix valuation argument in Section 2 gives the p -valuation axioms. If $\omega_a(g) > p/(p-1)$, division of (4.1) by p preserves the trace-zero condition and lowers depth by one. Its exponential is therefore a root in U^a . The coordinate conditions are closed, which gives completeness. $\square$

This replaces the integral-basis decomposition in [Pe, Proposition 3.6 and the subsequent saturation argument] when only saturation of the root group is required. For odd p , it also gives the root lattice comparison in [Pe, Claim 6.15] directly. More precisely, for $r \geq 0$ it identifies

$$\{u(x, y) : v(x) \geq r - a, v(y) \geq r - 2a\}$$

with the matrices in (4.2) satisfying $v(x) \geq r - a$, $v(z) \geq r - 2a$ and $z + \bar{z} = 0$. The same assertion holds for $p = 2$ when $r > 1$. No trace-zero integral basis is needed.

Example 4.2. The root in Proposition 4.1 can also be written down without logarithms. Multiplying matrices gives $u(x, y)u(x', y') = u(x + x', y + y' + x\bar{x}')$, and therefore

$$u(x, y)^k = u\left(kx, ky + \binom{k}{2}x\bar{x}\right), \quad u(x, y)^{1/p} = u\left(\frac{x}{p}, \frac{y}{p} - \frac{(p-1)x\bar{x}}{2p^2}\right)$$

for every integer $k \geq 1$. For example, let $p = 3$ and $E = \mathbb{Q}_3(i)$ with $i^2 = -1$, which is unramified over $F = \mathbb{Q}_3$. Then

$$\begin{pmatrix} 1 & 3 & 9/2 + 9i \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 9 & 81/2 + 27i \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $a = 0$, the right side has depth $2 > 3/2$, and its cube root on the left has depth 1.

5. TWO CORRECTIONS TO THE EXPLICIT FORMULAS

The two corrections below concern the arXiv version [Pe]. They do not give counterexamples to the existence of the saturated valuation or to the filtered Lie algebra isomorphism.

5.1. Rounding after ramified extension. In [Pe, Claim 6.14, pp. 42–43], let L/K be the splitting field used in the proof and let $r > 1/(p-1)$. The preceding remark defines $n_r = \lceil e_K r \rceil$, where e_K is the absolute ramification index. Both descent diagrams use the exponent $n_r e(L/K)$. At depth r , the split torus over L instead uses

$$(5.1) \quad \lceil e_L r \rceil = \lceil e(L/K) e_K r \rceil,$$

which need not equal $e(L/K) \lceil e_K r \rceil$.

For example, take $K = \mathbb{Q}_5$, $L = K(\pi)$ with $\pi^2 = 5$, and $T = \mathrm{Res}_{L/K} \mathbb{G}_m$. This torus is induced, so it satisfies the hypothesis in that claim. At $r = 1/2 > 1/4$, the standard filtration is

$$T(K)_{1/2} = 1 + \mathfrak{m}_L,$$

whereas the invariant group in the displayed diagram is $1 + \mathfrak{m}_L^2$. Thus $1 + \pi$ belongs to the required group but is omitted by the diagram. The corresponding Lie lattice omits π for the same reason. Replacing the exponent by (5.1) removes this discrepancy. The final identification with the integral torus model must then use the filtration at that corrected depth.

There is a convenient way to state the logarithm comparison without rounding. For a torus T split by a finite Galois extension L/K , its standard character filtration is

$$T(K)_r^{\mathrm{std}} = \{t \in T(K) : v(\chi(t) - 1) \geq r \text{ for every } \chi \in X^*(T)\}.$$

For $r > 1/(p-1)$, the scalar logarithm in split coordinates is injective and has image

$$\log(T(K)_r^{\mathrm{std}}) = \{X \in \mathrm{Lie}(T)(K) : v(d\chi(X)) \geq r \text{ for every } \chi \in X^*(T)\}.$$

To check this, choose split coordinates over L . Scalar logarithms preserve valuation on the indicated disk and commute with products, integral powers and Galois action. Applying the inverse exponential gives the inverse map. This argument works for every torus and every prime. The weakly induced assumption in [Pe] is used to compare the standard filtration with the minimal congruent filtration.

5.2. The conjugate root vectors. In [Pe, Remark 6.16, p. 44], the first two formulas for a root of type R2 omit the conjugate absolute root vector. Write $\tilde{\beta} = \tilde{\alpha} + \tilde{\alpha}'$ for the sum of the two conjugate absolute roots, and let $K_{\tilde{\alpha}}$ and $K_{\tilde{\beta}}$ be the fields of definition of $\tilde{\alpha}$ and $\tilde{\beta}$. The roots $\tilde{\alpha}$ and $\tilde{\alpha}'$ are conjugate over $K_{\tilde{\beta}}$. For $\Theta = \mathrm{Gal}(\bar{K}/K)$, let $\Theta_{\tilde{\beta}}$ be the stabilizer of $\tilde{\beta}$. The coefficients x, y, z belong to $K_{\tilde{\beta}}$, and $\gamma \in K_{\tilde{\alpha}}$ is the trace-zero element used in the paper. Here $X_{\tilde{\alpha}}$ and $X_{\tilde{\alpha}'}$ are its Chevalley–Steinberg root vectors. Differentiating the three coordinate subgroups in the rank-one model gives

$$(5.2) \quad x(E_{12} + E_{23}), \quad y\gamma(E_{12} - E_{23}), \quad z\gamma E_{13}.$$

The first two formulas in the paper should therefore read

$$\begin{aligned} \log u_{\alpha}(x, x^2/2) &\longmapsto \sum_{\sigma \in \Theta/\Theta_{\tilde{\beta}}} \sigma(x(X_{\tilde{\alpha}} + X_{\tilde{\alpha}'})), \\ \log u_{\alpha}(y\gamma, -y^2\gamma^2/2) &\longmapsto \sum_{\sigma \in \Theta/\Theta_{\tilde{\beta}}} \sigma(y\gamma(X_{\tilde{\alpha}} - X_{\tilde{\alpha}'})). \end{aligned}$$

Its third formula is unchanged. To see the problem in the printed expressions, it suffices to take the rank-one case $K_{\tilde{\beta}} = K$. The quotient indexing the sum then has one coset. Choosing the identity as representative gives $xX_{\tilde{\alpha}}$, while choosing the quadratic involution gives $xX_{\tilde{\alpha}'}$. These are different for $x \neq 0$. The corrected summands are invariant under this involution, so the sums are well defined. Formula (5.2) also follows immediately from (4.1).

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