

MIXED PRODUCTS OF STRONGLY PSEUDOPERFECT NUMBERS

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ABSTRACT. A number n is strongly pseudoperfect if some set of its divisors is closed under $d \mapsto n/d$ and adds up to $2n$. I show that two such numbers a and b with representations containing both endpoints can be mixed into a^2b , even when they are not coprime. This answers Questions 18 and 20 of Wang and Zelinsky. Separate constructions give Euler-form examples on all three possible sets of three odd prime factors. These numbers are not perfect powers, and are not divisible by the square of any strongly pseudoperfect number. I also count mixed-product examples and give two shorter proofs of earlier results.

1. INTRODUCTION

Let $D(n)$ be the set of positive divisors of n , and write $\sigma(n) = \sum_{d|n} d$. An integer $n > 1$ is *strongly pseudoperfect* if there is a set $S \subseteq D(n)$ such that

$$(1.1) \quad \{n/d : d \in S\} = S, \quad \sum_{d \in S} d = 2n.$$

So S is a set of divisors that comes in pairs d and n/d and adds up to exactly $2n$. When such a set can be chosen with $1, n \in S$, the integer is called *extremely strongly pseudoperfect*, and I write $\mathcal{E}$ for the set of these integers. Everything in this paper happens inside $\mathcal{E}$.

Wang and Zelinsky proved that all positive powers of an integer in $\mathcal{E}$ again belong to $\mathcal{E}$ [WZ, Theorem 5]. Their Questions 18 and 20 ask whether there are infinitely many primitive odd strongly pseudoperfect numbers, and whether infinitely many of them can have Euler's form. Here *primitive* means that the integer is not a power m^k of a strongly pseudoperfect integer m with $k \geq 2$. Euler's form means

$$(1.2) \quad n = p^e m^2, \quad p \equiv e \equiv 1 \pmod{4}, \quad p \text{ prime}, \quad m \text{ odd}, \quad p \nmid m.$$

All question and theorem numbers cited from [WZ] refer to version 2, dated 2 October 2026.

My main observation is that one can mix two different integers in $\mathcal{E}$, and they do not have to be coprime.

Theorem 🍌. *If $a, b \in \mathcal{E}$, then $a^2b \in \mathcal{E}$.*

The nice thing is that one prime exponent can stay small while the other exponents grow. A useful extra seed is $972405 = 3^4 \cdot 5 \cdot 7^4$, for which I give a divisor certificate in Section 4. Mixing it with $11025 = 3^2 \cdot 5^2 \cdot 7^2$ answers both questions at once.

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Theorem 🐼. *For every integer $k \geq 0$, the integer*

$$3^{12k+4}5^{12k+1}7^{12k+4}$$

is in $\mathcal{E}$, has Euler's form, and is not a nontrivial perfect power.

So all these integers are primitive. Three distinct prime factors is also the smallest number an odd strongly pseudoperfect integer can have. Mixing other seeds from [WZ] in the same way gives, for each r with $3 \leq r \leq 8$, infinitely many primitive odd members of $\mathcal{E}$ with exactly r distinct prime factors (Proposition 4.1).

In a discussion of an earlier version, Zelinsky suggested also excluding numbers obtained from the mixed-product construction [Z]. The constructions in Section 3 meet this stronger requirement on all three possible supports of three odd primes, namely 3, 5, 7, 3, 5, 11 and 3, 5, 13. Every member has Euler's form and is not a nontrivial perfect power, and no square of a strongly pseudoperfect integer divides it.

Following Zelinsky's comment in [Z], I present the counting bounds as consequences of the mixed-product construction. In Section 5, I count an explicit three-parameter family and obtain bounds of order $(\log x)^3$ for primitive Euler-form examples supported on 3, 5, 7. A separate four-parameter family in Section 6 keeps the distinguished prime equal to 29 with exponent 1. In Section 7, I give a binary proof of [WZ, Theorem 6] and an explicit divisor-set proof of [WZ, Theorem 9].

2. THE MIXED-PRODUCT CONSTRUCTION

In this section, I prove Theorem 🐼 by gluing together three blocks of divisors that live in three different intervals. For $n > 1$, membership in $\mathcal{E}$ is the same as the existence of a set

$$(2.1) \quad T_n \subseteq \{d \mid n : 1 < d < n\}, \quad \{n/d : d \in T_n\} = T_n, \quad \sum_{d \in T_n} d = n - 1.$$

To pass between the two descriptions, remove or add the endpoints 1, n . The condition $n > 1$ makes sure that the endpoints are distinct.

Proof of Theorem 🐼. Choose T_a, T_b as in (2.1), and put $N = a^2b$. I claim that

$$(2.2) \quad T_N = T_a \sqcup aT_b \sqcup abT_a$$

is a suitable set, where $aT_b = \{ad : d \in T_b\}$. Each element divides N , and the three blocks lie respectively in the disjoint intervals

$$(1, a), \quad (a, ab), \quad (ab, a^2b).$$

Reciprocity exchanges the first and third blocks, since

$$\frac{N}{d} = ab \frac{a}{d} \quad (d \in T_a).$$

It preserves the middle block because $N/(ae) = a(b/e)$ for $e \in T_b$. Finally,

$$\sum_{d \in T_N} d = (a - 1) + a(b - 1) + ab(a - 1) = a^2b - 1.$$

Thus (2.1) holds for N . □

The three intervals are the whole reason why a and b do not need to be coprime.

Corollary 2.1. *If $b, a_1, \dots, a_r \in \mathcal{E}$ and integers $e_1, \dots, e_r \geq 0$, then*

$$b \prod_{i=1}^r a_i^{2e_i} \in \mathcal{E}.$$

Proof. Apply [Theorem 2.1](#) repeatedly, once for each squared factor. $\square$

The power construction also has a short proof in this language. Namely, if $a \in \mathcal{E}$, then

$$T_{a^2} = T_a \sqcup aT_a$$

satisfies (2.1). Reciprocity exchanges the two blocks, and their sum is $(a-1)+a(a-1) = a^2-1$. Starting with a and a^2 , [Theorem 2.1](#) gives all positive powers of a .

3. EXAMPLES BEYOND THE MIXED PRODUCT

Zelinsky suggested excluding powers and mixed products a^2b in a stronger notion of primitivity [Z]. The next constructions meet this requirement on every possible set of three odd prime factors. The parameters in the following table are integers.

Theorem 3.1. *The following numbers belong to $\mathcal{E}$ and have Euler's form.*

prime factors	number	range	first member
3, 5, 7	$5 \cdot 7^2 \cdot 3^{2m}$	$m \geq 4$	1607445
3, 5, 7	$5 \cdot 3^4 \cdot 7^{4k}$	$k \geq 1$	972405
3, 5, 11	$3^4 \cdot 5^{4k+5} \cdot 11^2$	$k \geq 0$	30628125
3, 5, 13	$13 \cdot 5^4 \cdot 3^{2m}$	$m \geq 3$	5923125

No member is a perfect power or divisible by the square of a strongly pseudoperfect integer.

Proof. For the first row, write $n = 245 \cdot 3^{2m}$. Its exponent at 5 is 1, so it has Euler's form and is not a nontrivial perfect power. If $a^2 \mid n$, then the only possible prime factors of a are 3 and 7, and hence

$$\frac{\sigma(a)}{a} < \frac{3}{2} \frac{7}{6} = \frac{7}{4} < 2.$$

Thus a cannot be strongly pseudoperfect. It remains to construct the divisor representation.

I first give an extension that replaces m by $m+4$. Put

$$P = \{1, 5, 7, 35\},$$

$$Q = 3\{1, 5, 7, 49\} \sqcup 9\{1, 5, 7, 49, 245\} \sqcup 27\{5, 7\} \sqcup 81\{7\}.$$

The four parts of Q have different exponents at 3. Their sums, and the reciprocal sums needed below, are

$$(3.1) \quad \sum_{d \in P} d = 48, \quad \sum_{d \in P} \frac{245}{d} = 336, \quad \sum_{d \in Q} d = 3840, \quad \sum_{d \in Q} \frac{245 \cdot 81}{d} = 12320.$$

Suppose S is a representation of n whose elements not divisible by 3 are exactly P . For a divisor set A , write $n/A = \{n/d : d \in A\}$. With $N = 81^2 n$, define

$$(3.2) \quad S' = 81(S \setminus (P \cup n/P)) \sqcup P \sqcup N/P \sqcup Q \sqcup N/Q.$$

All displayed elements divide N . The five parts have exponents at 3 in the disjoint ranges

$$[5, 2m+3], \quad \{0\}, \quad \{2m+8\}, \quad [1, 4], \quad [2m+4, 2m+7].$$

Reciprocity preserves the first part and exchanges P with N/P and Q with N/Q . The endpoints are present, and (3.1) gives

$$\begin{aligned} \sum_{d \in S'} d &= 81 \left(2n - 48 - \frac{336n}{245} \right) + 48 + \frac{336N}{245} + 3840 + \frac{12320N}{245 \cdot 81} \\ &= 2N. \end{aligned}$$

The elements of S' not divisible by 3 are again exactly P , so the extension can be repeated.

For the four starting cases $m = 4, 5, 6, 7$, take $U_0 = P$ and the sets below. The last row gives U_m , and dashes mark unused rows.

	$m = 4$	$m = 5$	$m = 6$	$m = 7$
U_1	$\{1, 5, 35\}$	$\{1, 5, 7, 49\}$	$\{1, 5, 35\}$	$\{1, 5, 35\}$
U_2	$\{1, 5, 7, 35, 245\}$	$\{1, 5, 7\}$	$\{1, 5, 7, 49\}$	$\{1, 5, 7, 49\}$
U_3	$\{1, 5, 7\}$	$\{5\}$	$\{1, 5, 7, 35\}$	$\{1, 5\}$
U_4	—	$\{1, 5, 49, 245\}$	$\{1, 5, 35, 49, 245\}$	$\{1, 5, 7\}$
U_5	—	—	$\{7, 35\}$	$\{1, 49, 245\}$
U_6	—	—	—	$\{1, 7, 35\}$
U_m	$\{1, 7, 35, 245\}$	$\{7, 35\}$	$\{5, 49\}$	$\{1, 5, 7, 35, 49, 245\}$

Define

$$S = \bigcup_{i=0}^{m-1} \left\{ 3^i d, 3^{2m-i} \frac{245}{d} : d \in U_i \right\} \sqcup 3^m U_m.$$

The elements of S are distinct divisors of n by their exponents at 3. Each U_m is fixed by $d \mapsto 245/d$, so S is reciprocal and contains 1, n . Its sum can be checked from the table below.

m	4	5	6	7
$3^{-m} \sum_{i < m} 3^i \sum_{d \in U_i} d$	39	102	54	48
$490 \cdot 3^m - \sum_{i < m} 3^{m-i} \sum_{d \in U_i} \frac{245}{d}$	327	144	108	390
$\sum_{d \in U_m} d$	288	42	54	342

In every column, the first and third entries add up to the second. It follows that $\sum_{d \in S} d = 490 \cdot 3^{2m} = 2n$. The extension now proves the result for all $m \geq 4$.

For the second row, put $n = 405 \cdot 7^{4k}$. As in the first row, its exponent at 5 is 1, and every square divisor has a square root supported on 3 and 7. This gives Euler's form and both primitivity claims.

Define the lower divisors by

$$\begin{aligned} L &= \{1, 3, 5, 9, 15, 135\} \sqcup 7^{2k-1} \{1, 3, 5, 9, 15, 81, 135\} \\ &\sqcup \bigsqcup_{j=0}^{k-2} 7^{2j+1} \left(\{1, 3, 5, 9, 15, 27\} \sqcup 7 \{1, 5, 15, 135\} \right). \end{aligned}$$

Summing two geometric series gives

$$\sum_{d \in L} d = 39 \cdot 7^{2k}, \quad \sum_{d \in L} \frac{405}{d} = 810 - 679 \cdot 7^{-2k}.$$

The union over j is empty when $k = 1$. Now take

$$S = L \sqcup n/L \sqcup 7^{2k} \{1, 3, 9, 15, 27, 45, 135, 405\}.$$

The three parts have disjoint exponents at 7, respectively below, above and equal to $2k$. The last set in braces is fixed by $d \mapsto 405/d$ and has sum 640, so S is reciprocal and contains 1, n . Finally,

$$\sum_{d \in S} d = 39 \cdot 7^{2k} + \frac{n}{405} (810 - 679 \cdot 7^{-2k}) + 640 \cdot 7^{2k} = 2n.$$

Thus $n \in \mathcal{E}$.

For the third row, I give a representation of $n = 9801 \cdot 5^e$ for every integer $e \geq 5$. Let L consist of the divisors $5^i d$, where $d \mid 9801$ and $0 \leq i \leq 2$, except for the following values of d in each row:

i	omitted values of d
0	99, 121, 297, 891
1	11, 81, 121, 3267
2	11, 27, 121, 1089

Put $C = \{1, 3, 11, 27, 363, 891, 3267, 9801\}$. This set is fixed by $d \mapsto 9801/d$, and direct addition gives

$$\sum_{d \in L} d = 448875, \quad \sum_{d \in L} \frac{9801 \cdot 5^5}{d} = 60807375, \quad \sum_{d \in C} d = 14364.$$

Now take

$$S = L \sqcup n/L \sqcup \bigsqcup_{i=3}^{e-3} 5^i C,$$

where the last union is empty when $e = 5$. All entries are distinct divisors of n , since their exponents at 5 lie respectively in $[0, 2]$, $[e - 2, e]$ and $[3, e - 3]$. Reciprocity exchanges the first two parts and sends $5^i C$ to $5^{e-i} C$. The endpoints are present, and a geometric sum gives

$$\begin{aligned} \sum_{d \in S} d &= 448875 + 60807375 \cdot 5^{e-5} + 14364 \frac{5^{e-2} - 5^3}{4} \\ &= 61256250 \cdot 5^{e-5} = 2n. \end{aligned}$$

For $e = 4k + 5$, the number has Euler's form. Its prime exponents have greatest common divisor $\gcd(4, 4k + 5, 2) = 1$, so it is not a nontrivial perfect power. Finally, if $a^2 \mid n$, then the exponent of 3 in a is at most 2, and

$$\frac{\sigma(a)}{a} < \frac{13}{9} \frac{5}{4} \frac{11}{10} = \frac{143}{72} < 2.$$

Thus a cannot be strongly pseudoperfect.

For the last row, put $n = 8125 \cdot 3^{2m}$ with $m \geq 3$. The exponent of 13 is 1, and $13 \equiv 1 \pmod{4}$, so the number has Euler's form and is not a nontrivial perfect power. If a^2 divides it, then a is supported on 3, 5 and $\sigma(a)/a < 15/8 < 2$. Thus a cannot be strongly pseudoperfect.

I use the extension (3.2) again. This time put $P = D(8125) \setminus \{1625\}$, and let Q consist of $3^i d$ for $d \mid 8125$ and $1 \leq i \leq 4$, except for the following values of d in each row:

i	omitted values of d
1	65, 625
2	65, 1625, 8125
3	5, 325, 625, 1625, 8125
4	5, 65, 625, 1625

Direct addition gives

$$\sum_{d \in P} d = 9309, \quad \sum_{d \in P} \frac{8125}{d} = 10929, \quad \sum_{d \in Q} d = 744720, \quad \sum_{d \in Q} \frac{8125 \cdot 81}{d} = 425680.$$

Suppose S represents $n = 8125 \cdot 3^{2m}$ with $m \geq 3$ and its elements not divisible by 3 are exactly P . The five parts of (3.2) have the same disjoint ranges of exponents at 3 as before. They are reciprocal and contain the endpoints. The two equalities

$$744720 = 80 \cdot 9309, \quad 80 \cdot 10929 + 425680 = 160 \cdot 8125$$

give $\sum_{d \in S'} d = 2N$ by the calculation used for the first row. Thus this extension again replaces m by $m + 4$ and preserves the set P of divisors not divisible by 3.

It remains to give starting representations for $m = 3, 4, 5, 6$. In the following table only, write $D = D(8125)$. Set $U_0 = P$ and $U_1 = U_m = D$, and choose the remaining rows as follows:

	$m = 3$	$m = 4$	$m = 5$	$m = 6$
U_2	$\{1, 5, 625, 1625, 8125\}$	$D \setminus \{13, 25, 1625\}$	$D \setminus \{325\}$	$D \setminus \{8125\}$
U_3	—	$D \setminus \{625, 8125\}$	$\{1, 5, 65, 125\}$	$D \setminus \{325, 625\}$
U_4	—	—	$D \setminus \{1, 625\}$	$D \setminus \{1, 325, 1625\}$
U_5	—	—	—	$\{5\}$

As before, take the lower rows $3^i U_i$ for $i < m$, their reciprocals in n , and the middle row $3^m D(8125)$. All entries are distinct divisors by their exponents at 3. The middle row is reciprocal, and the endpoints are present. Their sums are checked by

m	3	4	5	6
$3^{-m} \sum_{i < m} 3^i \sum_{d \in U_i} d$	5020	2278	4024	1462
$16250 \cdot 3^m - \sum_{i < m} 3^{m-i} \sum_{d \in U_i} \frac{8125}{d}$	15954	13212	14958	12396

The second row exceeds the first by $10934 = \sigma(8125)$ in every column. Thus the full sum is $16250 \cdot 3^{2m} = 2n$. These four starting cases and the extension prove the result for every $m \geq 3$. The first member is 5923125. $\square$

Thus none of these examples is $a^2 b$ with a strongly pseudoperfect and b any positive integer. Exact searches through base-3 and base-7 carries led to these constructions. A search for a single repeated middle row with four or five lower boundary rows on the support 3, 5, 13 did not produce a certificate. This failed search is not a nonexistence result.

4. DIVISOR CERTIFICATES AND PRIMITIVE FAMILIES

Here are the small certificates I need for the constructions. If $O \subseteq D(n)$ is fixed by $d \mapsto n/d$, contains neither endpoint, and has sum $\sigma(n) - 2n$, then $D(n) \setminus O$ is a representation proving $n \in \mathcal{E}$. The certificates in this paper were also checked with exact integer arithmetic on a computer.

Put

$$(4.1) \quad \begin{aligned} u &= 11025 = 3^2 5^2 7^2, & v &= 23625 = 3^3 5^3 7, \\ z &= 42525 = 3^5 5^2 7, & c &= 972405 = 3^4 5^7 7^4. \end{aligned}$$

The seeds u, v, z , with the omitted divisors below, occur in [WZ, Table 1]. Notice that c has Euler's form with distinguished prime 5, and an exhaustive search over all integers $3^i 5^j 7^k$ below $3 \cdot 10^6$ shows that c is the smallest member of $\mathcal{E}$ of Euler's form with these three prime factors. The divisor sums are

n	$\sigma(n)$	$\sigma(n) - 2n$
u	22971	921
v	49920	2670
z	90272	5222
c	2033526	88716

and the omitted sets are

$$\begin{aligned} O_u &= \{25, 441, 35, 315, 105\}, \\ O_v &= \{21, 1125, 45, 525, 105, 225, 125, 189, 135, 175\}, \\ O_z &= \{27, 1575, 35, 1215, 45, 945, 63, 675, 75, 567\}, \\ O_c &= \{35, 27783, 45, 21609, 49, 19845, \\ &\quad 105, 9261, 135, 7203, 441, 2205\}. \end{aligned}$$

In each row, consecutive entries form a reciprocal pair, except for $105 \in O_u$, which is its own partner since $105^2 = u$. For example, the certificate for c reads

$$\begin{aligned} 35 \cdot 27783 &= 45 \cdot 21609 = 49 \cdot 19845 = 105 \cdot 9261 = 135 \cdot 7203 = 441 \cdot 2205 = c, \\ 35 + 27783 + 45 + 21609 + 49 + 19845 \\ &+ 105 + 9261 + 135 + 7203 + 441 + 2205 = 88716. \end{aligned}$$

All entries in each omitted set are distinct and lie strictly between 1 and the corresponding integer. These identities prove that all four seeds belong to $\mathcal{E}$.

Proof of Theorem 2.1. By Corollary 2.1, the integer cu^{6k} belongs to $\mathcal{E}$. Its factorization is

$$cu^{6k} = 3^{12k+4} 5^{12k+1} 7^{12k+4} = 5^{12k+1} \left(3^{6k+2} 7^{6k+2} \right)^2.$$

This has Euler's form because $12k + 1 \equiv 5 \equiv 1 \pmod{4}$. The prime exponents have greatest common divisor

$$\gcd(12k + 4, 12k + 1) = \gcd(12k + 1, 3) = 1.$$

An integer is a nontrivial perfect power exactly when its prime exponents have a common divisor greater than 1, so every member of the family is primitive. The integers are strictly increasing as k grows. $\square$

This proves affirmative answers to Questions 18 and 20 of [WZ]. Notice that the proof gives the stronger property that no member is a nontrivial perfect power.

For comparison, the first two seeds give another simple primitive odd family,

$$vu^{2k} = 3^{4k+3}5^{4k+3}7^{4k+1} \quad (k \geq 0).$$

Here the greatest common divisor of the exponents is $\gcd(4k+3, 2) = 1$.

Every odd strongly pseudoperfect integer has at least three distinct prime factors. If an odd integer $n > 1$ has at most two distinct prime factors, then

$$\frac{\sigma(n)}{n} < \prod_{p|n} \frac{p}{p-1} \leq \frac{3}{2} \frac{5}{4} = \frac{15}{8} < 2.$$

No subset of its divisors can then have sum $2n$, so the family in [Theorem 4.1](#) has the smallest possible number of prime factors, also among examples of Euler's form. Notice that the same estimate restricts the primes themselves, since $\prod_{p|n} p/(p-1) > 2$ forces the prime factors of an odd strongly pseudoperfect integer with exactly three of them to be 3, 5, 7, 3, 5, 11 or 3, 5, 13.

Other seeds from [WZ] can be mixed in the same way. All omitted sets displayed in [WZ, Table 1] avoid the endpoints 1 and n , so the ten integers of that table belong to $\mathcal{E}$. I use three of them, namely

$$12285 = 3^3 \cdot 5 \cdot 7 \cdot 13, \quad 16065 = 3^3 \cdot 5 \cdot 7 \cdot 17, \quad 45045 = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13.$$

I also use $937365 = 3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 23$, which is mentioned in [WZ] as a squarefree strongly pseudoperfect number. It belongs to $\mathcal{E}$, since the five reciprocal pairs

$$(55, 17043), \quad (57, 16445), \quad (65, 14421), \quad (143, 6555), \quad (165, 5681)$$

consist of distinct divisors strictly between the endpoints and have total sum $60630 = 1935360 - 2 \cdot 937365 = \sigma(937365) - 2 \cdot 937365$.

Proposition 4.1. *For each r with $3 \leq r \leq 8$, there are infinitely many primitive odd integers in $\mathcal{E}$ with exactly r distinct prime factors.*

Proof. For $r = 3$ this is [Theorem 4.1](#). For the other values of r , the integers

r	integers	k	prime
4	$12285 \cdot u^{2k}$	$k \geq 0$	13
5	$45045 \cdot u^{2k}$	$k \geq 0$	11
6	$45045 \cdot 16065^{2k}$	$k \geq 1$	11
7	$937365 \cdot u^{2k}$	$k \geq 1$	19
8	$937365 \cdot 16065^{2k}$	$k \geq 1$	19

belong to $\mathcal{E}$ by [Corollary 2.1](#), are odd, and have exactly r distinct prime factors for the stated k . In each row, the prime in the last column divides the first factor exactly once and does not divide the second factor. Its exponent in the product is therefore 1, so the product is not a nontrivial perfect power and in particular primitive. The integers in each row are strictly increasing in k . $\square$

The mixed product can never leave the union of the prime supports of its seeds, so larger values of r need odd seeds with more prime factors.

5. COUNTING EXAMPLES ON THREE PRIMES

I now count a three-parameter family built from the same seeds. Put

$$(5.1) \quad \mathcal{F} = \{cu^{2r}v^{4s}z^{2t} : r, s, t \in \mathbb{Z}_{\geq 0}\}.$$

Every member belongs to $\mathcal{E}$ by [Corollary 2.1](#). Its exponents at 3, 5, 7 are respectively

$$(5.2) \quad \begin{aligned} A &= 4 + 4r + 12s + 10t, \\ B &= 1 + 4r + 12s + 4t, \\ C &= 4 + 4r + 4s + 2t. \end{aligned}$$

Thus A, C are even and $B \equiv 1 \pmod{4}$, so all members have Euler's form with distinguished prime 5. The parameters determine distinct integers because

$$(5.3) \quad \det \begin{pmatrix} 4 & 12 & 10 \\ 4 & 12 & 4 \\ 4 & 4 & 2 \end{pmatrix} = -192 \neq 0.$$

Let $F_0(x)$ count the members of $\mathcal{F}$ at most x that are not nontrivial perfect powers. Equivalently, it counts the parameter triples for which $\gcd(A, B, C) = 1$.

Theorem 5.1. *As $x \rightarrow \infty$, we have*

$$F_0(x) = \frac{(\log(x/c))^3}{91\zeta(3)\log u \log v \log z} + O((\log x)^2).$$

Here $\zeta(3) = \sum_{j \geq 1} j^{-3}$. The exact formula counts members that are not perfect powers. It gives a lower bound for primitive members in the sense of [\[WZ\]](#), since that definition may also allow some perfect powers.

First I record the elementary lattice estimate used in the proof.

Lemma 5.2. *Fix positive real numbers $\lambda_1, \lambda_2, \lambda_3$. Uniformly in a residue class modulo $d \geq 1$, the number of nonnegative integer triples satisfying $\lambda_1 r + \lambda_2 s + \lambda_3 t \leq L$ is*

$$\frac{L^3}{6\lambda_1\lambda_2\lambda_3d^3} + O((L/d + 1)^2) \quad (L \geq 0).$$

Proof. For an unrestricted simplex with bound $Y \geq 0$, place a unit cube at each nonnegative integer point. The union contains the simplex with bound Y and lies in the simplex with bound $Y + \lambda_1 + \lambda_2 + \lambda_3$. Comparing volumes gives

$$\#\{(r, s, t) \in \mathbb{Z}_{\geq 0}^3 : \lambda_1 r + \lambda_2 s + \lambda_3 t \leq Y\} = \frac{Y^3}{6\lambda_1\lambda_2\lambda_3} + O((Y + 1)^2).$$

For a fixed residue triple (k_1, k_2, k_3) with $0 \leq k_i < d$, write $(r, s, t) = (k_1, k_2, k_3) + d(i, j, \ell)$. The new bound is

$$Y = \frac{L - \lambda_1 k_1 - \lambda_2 k_2 - \lambda_3 k_3}{d} = L/d + O(1).$$

If $Y \geq 0$, the first estimate proves the claim. When $Y < 0$, the count is zero and L/d is bounded, so the same estimate holds. $\square$

Proof of Theorem 5.1. Write $L = \log(x/c)$. The parameter region is

$$(5.4) \quad 2r \log u + 4s \log v + 2t \log z \leq L, \quad r, s, t \geq 0.$$

Its volume is $L^3/(96 \log u \log v \log z)$. For a positive integer d , let $\rho(d)$ be the number of residue triples modulo d for which $d \mid A, B, C$.

There are no solutions modulo 2, because B is odd. Modulo 3, the equations are

$$1 + r + t = 0, \quad 1 + r + s + 2t = 0.$$

For each choice of t , these determine $r = -1 - t$ and $s = -t$, so there are three solutions. Modulo a prime $p \geq 5$, (5.3) shows that the coefficient matrix is invertible, giving exactly one solution. The Chinese remainder theorem therefore gives

$$(5.5) \quad \rho(d) = \begin{cases} 0, & 2 \mid d, \\ 3, & d \text{ odd and } 3 \mid d, \\ 1, & \gcd(d, 6) = 1 \end{cases} \quad (d \text{ squarefree}).$$

In particular, $\rho(d) \leq 3$ for squarefree d .

Throughout (5.4), the positive integer B is at most $K(L + 1)$ for a fixed constant $K > 0$. Thus $\gcd(A, B, C) \leq D$, where $D = \lfloor K(L + 1) \rfloor$. Let μ be the Möbius function. Using the identity

$$\mathbf{1}_{\gcd(A, B, C)=1} = \sum_{d \mid A, d \mid B, d \mid C} \mu(d)$$

and applying Lemma 5.2 to every relevant residue class gives

$$\begin{aligned} F_0(x) &= \frac{L^3}{96 \log u \log v \log z} \sum_{d \leq D} \frac{\mu(d) \rho(d)}{d^3} \\ &\quad + O \left(\sum_{d \leq D} |\mu(d)| \rho(d) (L/d + 1)^2 \right). \end{aligned}$$

The error is $O(L^2 + L \log(L + 2) + D) = O(L^2)$ as $L \rightarrow \infty$. Extending the main sum to infinity costs $O(L^3 D^{-2}) = O(L)$. Finally, (5.5) gives the absolutely convergent product

$$\begin{aligned} \sum_{d \geq 1} \frac{\mu(d) \rho(d)}{d^3} &= \left(1 - \frac{3}{3^3}\right) \prod_{p \geq 5} (1 - p^{-3}) \\ &= \frac{8}{9} \frac{1}{(1 - 2^{-3})(1 - 3^{-3})\zeta(3)} = \frac{96}{91\zeta(3)}. \end{aligned}$$

Multiplying by the simplex volume proves the formula. $\square$

Let $E(x)$ count the primitive strongly pseudoperfect integers at most x that have Euler's form and whose prime factors are exactly 3, 5, 7. Every integer counted by $F_0(x)$ is counted by $E(x)$, so Theorem 5.1 gives a positive lower bound of order $(\log x)^3$. For the upper bound, Euler's form forces even exponents at 3 and 7 and an exponent $\equiv 1 \pmod{4}$ at 5. The same simplex estimate counts $(\log x)^3/(96 \log 3 \log 5 \log 7) + O((\log x)^2)$ such candidates. Thus, for all sufficiently large x ,

$$(5.6) \quad C_1(\log x)^3 \leq E(x) \leq C_2(\log x)^3$$

whenever $0 < C_1 < 1/(91\zeta(3) \log u \log v \log z)$ and $C_2 > 1/(96 \log 3 \log 5 \log 7)$. This gives the largest possible order on the fixed prime support, without an exact asymptotic for all primitive strongly pseudoperfect integers on these primes.

6. A FAMILY WITH FIXED DISTINGUISHED EXPONENT

In this section, I come back to the extra requirement that the distinguished exponent stays equal to 1. Use

$$b = 1102725 = 3^2 5^2 13^2 \cdot 29, \quad w = 10661625 = 3^8 5^3 \cdot 13.$$

The integer b is an Euler-form example given in [WZ]. Both integers belong to $\mathcal{E}$ by the following certificates. For b , the omitted pairs are

$$(325, 3393), \quad (377, 2925).$$

Both products are b , and their total sum is $7020 = 2212470 - 2b = \sigma(b) - 2b$. For w , omit

$$(81, 131625), \quad (351, 30375), \quad (2187, 4875).$$

Their products are w , and their total sum is $169494 = 21492744 - 2w = \sigma(w) - 2w$. All these divisors are distinct and strictly between the corresponding endpoints.

Proposition 6.1. *For integers $r, s, h, t \geq 0$, the numbers $bu^{2r}v^{2s}z^{2h}w^{2t}$ are distinct primitive members of $\mathcal{E}$ of Euler's form, and*

$$\#\{bu^{2r}v^{2s}z^{2h}w^{2t} \leq x : r, s, h, t \geq 0\} = \frac{(\log(x/b))^4}{384 \log u \log v \log z \log w} + O((\log x)^3).$$

Proof. Membership follows from Corollary 2.1. Each integer is 29 times the square of an integer not divisible by 29, so it has Euler's form and cannot be a nontrivial perfect power. The multipliers u, v, z, w are multiplicatively independent. To see this, the 13-adic valuation first removes w from any multiplicative relation, and the remaining exponent vectors have determinant

$$\det \begin{pmatrix} 2 & 3 & 5 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix} = -12 \neq 0.$$

Thus the four parameters give distinct integers. Counting lattice points in the simplex

$$2r \log u + 2s \log v + 2h \log z + 2t \log w \leq \log(x/b)$$

gives the stated volume, since $4!2^4 = 384$. The unit-cube comparison from Lemma 5.2, applied in four dimensions, gives the cubic error. $\square$

So there are $\gg (\log x)^4$ primitive examples of Euler's form with the exponent of 29 fixed at 1.

7. TWO SHORTER PROOFS

In this section, I give alternative proofs of two results of Wang and Zelinsky. The results are theirs, and only the proofs are new.

7.1. The Mersenne-prime restriction. The following is [WZ, Theorem 6].

Proposition 7.1. *Let $k \geq 0$ and $m \geq 2$ be integers with $p = 2^m - 1$ prime. If $2^k p$ is strongly pseudoperfect, then $k = m - 1$.*

Proof. Put $n = 2^k p$. All reciprocal divisor pairs have the form $(2^i, p2^{k-i})$ with $0 \leq i \leq k$. Let I index the omitted pairs and put

$$B = \sum_{i \in I} 2^i, \quad B^* = \sum_{i \in I} 2^{k-i}.$$

The sum of the omitted divisors is

$$(7.1) \quad B + pB^* = \sigma(n) - 2n = 2^{k+1} - 2^m.$$

Nonnegativity gives $k+1 \geq m$, and equality gives the conclusion. Suppose that $k+1 > m$. Reducing (7.1) modulo 2^m gives $B \equiv B^* \pmod{2^m}$. The lowest m binary digits therefore agree, so

$$i \in I \iff k-i \in I \quad (0 \leq i < m).$$

If such an i belongs to I , the omitted sum contains both 2^{k-i} and $p2^{k-i}$, whose sum is

$$(1+p)2^{k-i} = 2^{m+k-i} \geq 2^{k+1}.$$

This exceeds the right-hand side of (7.1). Hence no element of I is smaller than m . If $i \in I$ and $k-i < m$, the equivalence applied to $k-i$ gives $k-i \in I$, which we just excluded. So every $i \in I$ satisfies $m \leq i \leq k-m$. If this interval is empty, the omitted sum is zero, which is impossible under $k+1 > m$. Otherwise,

$$B + pB^* \leq (1+p) \sum_{i=m}^{k-m} 2^i = 2^{k+1} - 2^{2m} < 2^{k+1} - 2^m.$$

This also contradicts (7.1). □

In the possible case, $2^k p = 2^{m-1}(2^m - 1)$ is perfect. The binary digits replace the separate cases concerning the smallest omitted power of 2. Notice that the hypothesis on p cannot be dropped, since $352 = 2^5 \cdot 11$ is strongly pseudoperfect with the omitted pair $(8, 44)$.

7.2. An explicit divisor set. The next construction is from [WZ, Theorem 9]. I make the endpoint condition explicit and check directly that the divisors stay distinct when the factors are composite.

Proposition 7.2. *Let $a > b \geq 1$ be integers, $q = 2^{b+1} - 1$, and $p = 2^{a+1} - 2^b - 1$. Then $pq2^a$ belongs to $\mathcal{E}$.*

Proof. Put $n = pq2^a$ and take

$$\begin{aligned} S = & \{2^j, q2^j : 0 \leq j \leq a, j \neq b\} \\ & \sqcup \{p2^j, pq2^j : 0 \leq j \leq a, j \neq a-b\}. \end{aligned}$$

The odd parts $1, q, p, pq$ are distinct because $q > 1$ and $p - q = 2^{a+1} - 3 \cdot 2^b \geq 2^b$. Thus all displayed divisors are distinct, even if p and q share prime factors. The map $d \mapsto n/d$ pairs the four types by replacing j with $a-j$, and the omitted indices are paired as well. Hence S is fixed by reciprocity and contains $1, n$. Finally,

$$\begin{aligned} \sum_{d \in S} d &= (q+1) \left[(2^{a+1} - 1 - 2^b) + p(2^{a+1} - 1 - 2^{a-b}) \right] \\ &= (q+1)p(2^{a+1} - 2^{a-b}) \\ &= 2^{b+1}p2^{a-b}(2^{b+1} - 1) = 2n. \end{aligned}$$

This proves $n \in \mathcal{E}$. □

8. WHAT IS STILL OPEN

The mixed-product construction answers Questions 18 and 20 of [WZ] with exactly three prime factors, and (5.6) gives the largest possible counting order on that fixed support. The families in Theorem 3.1 also survive the stronger restriction suggested in [Z]. Notice that all constructions give representations containing the endpoints. My arguments do not answer Question 19 about powers of an arbitrary strongly pseudoperfect integer, or Question 21 about positive natural density. They also say nothing about the existence of odd perfect numbers.

A natural next step is to let two prime exponents grow independently while preserving the stronger restriction. Searches using finite boundary strips and repeated middle rows did not find such a construction for $5 \cdot 3^{2a}7^{2b}$ or $3^4 \cdot 5^e11^{2b}$. These searches were limited to specific forms of the divisor sets and give no obstruction to the existence of such families.

AI & Taco disclosure. The AI tools used for this note were Astra 6 Ultra, Fable 5.1 and Opus 5.5. They did most of the mathematical research and writing, including developing the constructions and proofs, searching for examples and checking the arguments. The divisor certificates were also checked using exact integer arithmetic. I write under the name TacoYaci, a random Reddit user who is a professionally trained mathematician in real life. I release these results anonymously because no credit for the mathematical discoveries or proofs should be given to me as the human prompter. Wang and Zelinsky deserve credit for the results, examples and questions from which this note starts.

REFERENCES

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