

WHEN A NUMBER IS THE DIGIT SUM OF ITS OWN POWER

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ABSTRACT. I study positive integers satisfying $n = s(n^k)$, where s is the decimal digit sum. A congruence condition and a random digit model explain patterns in the solutions. The model suggests infinitely many pairs and triples of consecutive exponents. I conjecture that 90 is the only $n > 1$ with four consecutive exponents. In other bases, an explicit construction gives arbitrarily long runs.

1. INTRODUCTION

The number 26 is the sum of the digits of its own cube:

$$26^3 = 17576, \quad 1 + 7 + 5 + 7 + 6 = 26.$$

Writing $s(m)$ for the decimal digit sum of a positive integer m , I study the equation

$$(1.1) \quad n = s(n^k), \quad n, k \geq 1.$$

The solution $n = 1$ occurs for every exponent and will be called trivial. For $k = 3$ the six solutions are 1, 8, 17, 18, 26, 27. Their cubes are the classical Dudeney numbers. The general equation is also known as the generalized Dudeney problem [J, R].

Let $m(k)$ count the solutions for a fixed k . This number is finite, since

$$(1.2) \quad s(n^k) \leq 9(1 + k \log_{10} n),$$

whose right-hand side grows more slowly than n . See also [KN] for the analogous argument with a polynomial in place of a power. The OEIS entry [A046019](#) lists $m(k)$ through $k = 1000$, while [A124359](#) lists numbers that solve the equation for some exponent [OEIS].

The complete lists for $k \leq 4500$ contain 22002 pairs, including the trivial ones. The successive records of $m(k)$ are

k	1	13	25	619	703	2533	3223	4063
$m(k)$	9	12	14	15	16	18	19	21
$k \bmod 6$	1	1	1	1	1	1	1	1

Every record exponent after 1 is congruent to 1 modulo 6. The elementary explanation begins with the following restriction.

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Theorem 🍌. Let $k \geq 2$ and suppose $n = s(n^k)$.

- i) $n^k \equiv n \pmod{9}$. This congruence has exactly $1 + \gcd(k-1, 6)$ solutions modulo 9.
- ii) If also $n = s(n^{k+1})$, then $n \equiv 0$ or $1 \pmod{9}$.

Thus seven residue classes are available when $k \equiv 1 \pmod{6}$, and only two, three or four otherwise. This favours large values of $m(k)$ in that class. It does not prove that large values occur only there.

The same number can solve (1.1) at consecutive exponents. The most striking example is

$$(1.3) \quad 90 = s(90^{19}) = s(90^{20}) = s(90^{21}) = s(90^{22}).$$

Xoff displayed this quadruple and the triple 181 at exponents 18, 19, 20 in a public answer in 2015 [X]. These solutions also appear in public tables [H, OEIS]. No second quadruple occurs in the complete data through 4500 or in the further search described below. I propose the following.

Conjecture 1.1. The only positive integers $n > 1$ and k satisfying

$$n = s(n^k) = s(n^{k+1}) = s(n^{k+2}) = s(n^{k+3})$$

are $n = 90$ and $k = 19$.

I compare the data with a random digit model in Sections 3 and 4. It suggests unbounded values of $m(k)$ and infinitely many twins and triples, but only finitely many quadruples. These are heuristic predictions. In other bases there is an explicit construction. Write s_b for the digit sum in base b , and $v_2(n)$ for the exponent of 2 in n .

Theorem 🍌. Let $b \geq 3$ be odd and $n = (b+1)/2$. Then $n = s_b(n^k)$ for every $1 \leq k \leq 1 + v_2(n)$.

For $b = 2^a - 1$ and $n = 2^{a-1}$ this gives at least a consecutive exponents. Hence runs of arbitrary length exist when the base may vary. The proof is in Section 5.

2. THE CONGRUENCE CONDITION

Every positive integer satisfies $s(m) \equiv m \pmod{9}$, because $10 \equiv 1 \pmod{9}$.

Proof of Theorem 🍌. The equation gives $n^k \equiv n \pmod{9}$. If $3 \mid n$, then $9 \mid n^k$ because $k \geq 2$, so $9 \mid n$. Otherwise n is a unit modulo 9 and $n^{k-1} \equiv 1$. The units form a cyclic group of order 6, so there are $\gcd(k-1, 6)$ such classes, in addition to 0.

For two consecutive exponents, $n^k \equiv n^{k+1} \equiv n \pmod{9}$. Therefore $n(n-1) \equiv 0 \pmod{9}$. Since n and $n-1$ are coprime, one of them is divisible by 9. $\square$

Trailing zeros matter too. Write $n = 10^j m$ with $10 \nmid m$. Then

$$s(n^k) = s(m^k).$$

The equation becomes $10^j m = s(m^k)$. This explains the separate branches in Figure 1: the power of m , rather than that of n , determines how many digits contribute to the sum.

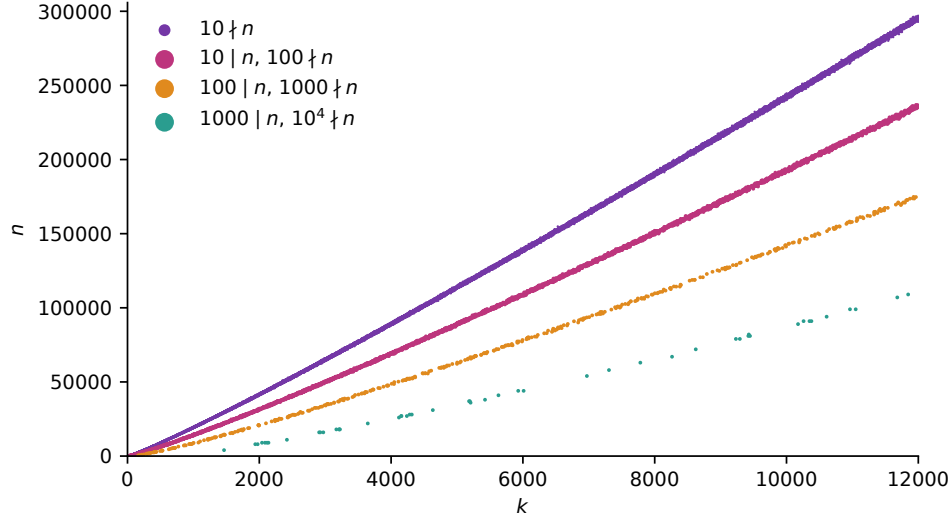


FIGURE 1. Solutions with $n > 1$ and $k \leq 12000$, coloured by the power of 10 dividing n . The lists are complete through 4500. Beyond that, the search uses a window.

3. A RANDOM DIGIT MODEL

Write $n = 10^j m$ with $10 \nmid m$ and $m \geq 2$. I approximate the number of digits of m^k by $D = k \log_{10} m$, ignoring the rounding and the extra digit. Independent uniform digits in $\{0, \dots, 9\}$ would give mean $4.5D$ and variance $8.25D$ for their sum. A normal approximation then suggests the weight

$$(3.1) \quad p_k(n) = \frac{9}{\sqrt{2\pi \cdot 8.25D}} \exp\left(-\frac{(n - 4.5D)^2}{2 \cdot 8.25D}\right)$$

when $n^k \equiv n \pmod{9}$, and $p_k(n) = 0$ otherwise. The factor 9 accounts for the known residue of the digit sum. Numbers with $m = 1$ have no nontrivial solutions and receive weight zero.

I treat these weights as approximate probabilities and assume independence for different n and different k . The approximation is intended for large digit sums. At $k = 1$ it even gives weights above one for some small n , so it is not a literal probability law there. The idea of modelling digits this way, including the congruence and trailing zeros, already appears in [D, R, S].

Under this model, $m(k) - 1$ is approximately Poisson with mean

$$(3.2) \quad \lambda(k) = \sum_{n \geq 2} p_k(n).$$

Most weight lies near $10^j m = 4.5k \log_{10} m$. For the numerical evaluations I sum over $j = 0, 1, 2, 3$ and all integers $m \geq 2$ with $10 \nmid m$ within twelve standard deviations of this mean. The following comparison includes $1 \leq k \leq 4500$.

$k \bmod 6$	Allowed classes	Mean $m(k) - 1$	Mean $\lambda(k)$
0	2	2.29	2.23
1	7	7.86	7.80
2	2	2.19	2.22
3	3	3.17	3.34
4	4	4.60	4.45
5	3	3.23	3.34

TABLE 1. Observed and predicted class means. The count of allowed residue classes applies to $k \geq 2$.

The congruence explains most of the difference between the classes. Within each class, the observed variance is also within about five percent of the mean, as the Poisson model suggests. The expected numbers of exponents $k \leq 4500$ with $m(k) \geq 21$ and $m(k) \geq 22$ are 0.143 and 0.0525. The data contain one of the former and none of the latter.

Extrapolating the same formula, the expected count for $m(k) \geq 22$ first reaches one at $K = 139933$. This is a scale for the search, not a predicted first exponent. The model suggests that $m(k)$ is unbounded in every class modulo 6, with the largest records usually coming from $k \equiv 1 \pmod{6}$.

4. RUNS OF CONSECUTIVE EXPONENTS

A run is a maximal string of consecutive exponents satisfying (1.1) for the same $n > 1$. I call runs of lengths two, three and four twins, triples and quadruples. For comparison with the model, I count blocks of r consecutive solutions without requiring maximality. The quadruple for 90 contains two triple blocks.

By Theorem 4.1, every block of length at least two has $n \equiv 0$ or $1 \pmod{9}$. The expected number of blocks of length r with starting exponent at most K is

$$(4.1) \quad \sum_{k=1}^K \sum_{n \geq 2} \prod_{i=0}^{r-1} p_{k+i}(n).$$

Each factor uses its own exponent. In particular, the mean moves by $4.5 \log_{10} m$ at each step.

Near the mean $n = 4.5D$, one further equality has approximate probability

$$\frac{9}{\sqrt{2\pi \cdot 11n/6}} \approx \frac{2.65}{\sqrt{n}}.$$

Thus further steps become less likely as n grows. The model gives the following counts, using the same branches and window as in Section 3.

Block length	Observed	Expected
2	120	134
3	5	4.90
4	1	0.379

TABLE 2. Blocks with starting exponent $1 \leq k \leq 4500$. The five triple blocks belong to four maximal runs.

The infinite sums behave differently for the three lengths. Summing first over exponents for a fixed $n = 10^j m$ gives a contribution of order

$$\frac{1}{n^{(r-1)/2} \log m}.$$

Summing this over the two admissible classes of m diverges for $r = 2, 3$ and converges for $r \geq 4$, separately on each branch. Equivalently, the expected counts up to exponent K grow on the scale $\sqrt{K/\log K}$ for pairs and $\log \log K$ for triples.

For quadruples the total expected count over all starting exponents is 0.383. Only 0.00473 comes from $k > 4500$. These are numerical predictions of the model restricted to $j \leq 3$. They motivate [Conjecture 1.1](#), but they cannot exclude a structured exception. Multiplication creates correlations between digit sums, and the independence assumption is only an approximation.

Could a second quadruple be constructed? Separated decimal blocks alone do not seem sufficient. For example, if every binomial coefficient in $(10^d + 1)^k$ is below 10^d , then $2^k < (k+1)10^d$. Hence $k = O(d)$ and the digit sum is $O(d^2)$, eventually smaller than $10^d + 1$. This rules out that construction while its blocks remain separated. It says nothing about constructions involving carries.

There is also a rigorous result for each fixed $n > 1$: only finitely many exponents satisfy $s(n^k) = n$. For powers of ten this is immediate. For the other n , Stewart's lower bound for the number of nonzero digits tends to infinity [[St](#), Theorem 2]. This does not give a uniform bound as n varies, so it does not settle the quadruple conjecture.

5. OTHER BASES

In base b , the necessary congruence is $n^k \equiv n \pmod{b-1}$. Consecutive exponents force $n(n-1) \equiv 0 \pmod{b-1}$. Each prime power dividing $b-1$ must therefore divide either n or $n-1$. More distinct prime factors give more admissible residue classes for runs.

There is also the explicit construction of [Theorem 4.1](#).

Proof of [Theorem 4.1](#). Let $n = (b+1)/2$ and $1 \leq k \leq 1 + v_2(n)$. Then $u = n/2^{k-1}$ is an integer, and

$$n^k = u(b+1)^{k-1} = \sum_{i=0}^{k-1} u \binom{k-1}{i} b^i.$$

Every coefficient is at most $u2^{k-1} = n \leq b-1$, so this is the base b expansion. Its digit sum is $u2^{k-1} = n$. $\square$

For example, 4 is the digit sum of its first three powers in base 7, and 16 of its first five powers in base 31. This construction needs $(b+1)/2$ to be an integer, so it does not apply in base 10.

For runs beginning at the first exponent there is another elementary observation.

Proposition 5.1. *Let $1 \leq n < b$. Then $n = s_b(n^2)$ if and only if $(b-1) \mid n(n-1)$.*

Proof. Necessity follows from $s_b(n^2) \equiv n^2 \pmod{b-1}$. For sufficiency, put $q = n(n-1)/(b-1)$. Then $0 \leq q \leq n-1$ and

$$n^2 = qb + (n-q), \quad 1 \leq n-q \leq b-1.$$

This is the base b expansion, with digit sum n . □

In decimal, the only nontrivial example at $k=1$ is $9 = s(9) = s(81)$.

The exhaustive computations for $2 \leq b \leq 36$ and $k \leq 600$ found 17 blocks of length four starting at $k \geq 2$, against 24.0 predicted by the corresponding model. No block of length five starts at $k \geq 2$ in these data. These finite data, together with the construction above, show that long runs depend strongly on the base.

6. THE COMPUTATIONS AND OPEN QUESTIONS

The solution lists for every $k \leq 4500$ are complete. An exhaustive computation tested all candidates allowed by (1.2) and the congruence modulo 9, using exact integer powers and digit sums. An independent recomputation found no discrepancies. For $k \leq 1000$ the lists also agree with [A152147](#) [OEIS].

For $4500 < k \leq 12000$, the search tested candidates on the branches $j = 0, \dots, 4$ within seven standard deviations of the model mean. Every reported pair was checked exactly, but this search does not prove completeness. It found one further triple, 286875 at exponents 11675, 11676, 11677, and no further quadruple. The maximal runs of length at least three found in the combined data are

n	k	length	exponents
181	18	3	$k = 18, \dots, 20$
90	19	4	$k = 19, \dots, 22$
17830	1224	3	$k = 1224, \dots, 1226$
30510	1955	3	$k = 1955, \dots, 1957$
286875	11675	3	$k = 11675, \dots, 11677$

The complete lists through 4500 and the additional solutions found through 12000 are supplied in the accompanying data file.

The main open questions remain simple. Is $m(k)$ unbounded? Are there infinitely many twins and triples? Is [Conjecture 1.1](#) true? The model gives suggested answers, but none is proved. In another fixed base, is the length of a run bounded? The construction in [Theorem 1](#) answers the question only when the base may vary.

AI & Taco disclosure. I write as TacoYaci. The equation, the original data through $k = 4500$ and the observations about records and runs come from my earlier work. I used Claude Code with Opus 5.5 and Fable 5.1, and Codex, for independent computations, the elementary proofs, the model, literature checks and drafting. The model is heuristic, and agreement with finite data is not a proof of its predictions.

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